Energy Spectrum of Quasi-Geostrophic Turbulence

Peter Constantin

Department of Mathematics

The University of Chicago

7/16/02

Abstract. We consider the energy spectrum of a quasi-geostrophic model of forced, rotating turbulent flow. We provide a rigorous a priori bound $E(k) \leq Ck^{-2}$ valid for wave numbers that are smaller than a wave number associated to the forcing injection scale. This upper bound separates this spectrum from the Kolmogorov-Kraichnan $k^{-\frac{5}{3}}$ energy spectrum that is expected in a two-dimensional Navier-Stokes inverse cascade. Our bound provides theoretical support for the k^{-2} spectrum observed in recent experiments.

The typical time scales associated with atmospheric flow over long distances are much bigger than the time scales associated with Earth's rotation. This low Rossby number situation is characterized by a relative suppression of momentum transfer across vertical scales, and the organization of the flow in quasi-two dimensional strata. Cyclonic and anti-cyclonic vortical motion ensues in these layers, with dynamics in which strong, interacting vortices of many sizes are born, grow, and dissipate, over time scales that are long compared to the rotation time scale. The precise mathematical way of describing such a quasi-two dimensional picture is yet unclear. Energy spectra are some of the most robust quantitative indicators that one can use in order to distinguish between different classes of models. If a strictly two dimensional Navier-Stokes framework is adopted for rotating turbulence then the predicted energy spectra are a k^{-3} direct enstrophy cascade (at wave numbers larger than the wave number of the forces stirring the fluid) and a Kolmogorov-Kraichnan $k^{-\frac{5}{3}}$ inverse energy cascade spectrum at wave numbers that are smaller than the forcing wave numbers [1]. Recent experiments [2] of rotating fluids find a different inverse energy cascade power spectrum: $E \sim k^{-2}$. This spectrum implies a steeper inverse energy cascade

than the one predicted by a strictly two-dimensional Kolmogorov-Kraichnan spectrum. The k^{-2} spectrum was observed for wave numbers of $10^{-1} - 10^{0}$ cm^{-1}. The experimental data showed the steeper k^{-2} spectrum clearly separated from a $k^{-\frac{5}{3}}$ spectrum that was fit to agree with $E(k_0)$ at the largest scale $k_0 \sim 10^{-1}$ cm^{-1}.

The purpose of this letter is to describe a rigorous upper bound $E(k) \leq Ck^{-2}$ valid in the inverse cascade region, in a quasi-geostrophic regime.

The most important feature of strongly rotating fluids is the geostrophic balance between the Coriolis force and pressure gradients. This balance, valid only in a first approximation, imposes a two-dimensional time independent solution. The departure from this balance, to lowest order, has non-trivial dynamics and is described by quasi-geostrophic equations [3], which are quasi-two dimensional equations asserting the conservation of potential vorticity q subject to dynamical boundary conditions. The simplest of these can be written as

$$\partial_t \theta + \mathbf{v} \cdot \nabla \theta + w_E \Lambda \theta = f \tag{1}$$

The two-dimensional velocity $\mathbf{v}$ is incompressible, $\nabla \cdot \mathbf{v} = 0$. The dissipative term $w_E \Lambda \theta$ has a coefficient $w_E > 0$ that comes from Ekman pumping at the boundary. This coefficient has units of velocity and, for the situations we consider, where a non-trivial vertical flux is imposed, this coefficient is not vanishingly small. (In the experiment ([2]) the vertical velocities at the boundary are close in magnitude to the maximal measured velocities, and are in ranges of about 20-30 cm/s; the value for w_E is expected to match the same order of magnitude). The operator Λ can be described in Fourier representation as multiplication by the magnitude k of the wave number $\mathbf{k} = (k_x, k_y)$, that is:

$$\widehat{\Lambda \theta}(\mathbf{k}, t) = k \widehat{\theta}(\mathbf{k}, t). \tag{2}$$

The velocity is related to θ by

$$\widehat{\mathbf{v}}(\mathbf{k}, t) = (-k_y, k_x) \frac{\sqrt{-1}}{k} \widehat{\theta}(\mathbf{k}, t) \tag{3}$$

Thus, for each wave number

$$|\widehat{\theta}(\mathbf{k}, t)| = |\widehat{\mathbf{v}}(\mathbf{k}, t)|. \tag{4}$$

The symbol f refers to the forcing term. This active scalar surface quasi-geostrophic equation has been studied both analytically and numerically [4].

We will analyze the energy spectrum using the Littlewood-Paley decomposition [5]. The Littlewood-Paley decomposition is not orthogonal, but it is nearly so. Its use affords great flexibility in dealing with functions that involve many active scales: wave numbers are grouped in dyadic blocks and averages over the dyadic blocks are performed. The Littlewood-Paley decomposition is defined in terms of a smooth partition of unity in Fourier space. This partition is constructed starting from a nonnegative, nonincreasing, radially symmetric function $\phi_{(0)}(\mathbf{k}) = \phi_{(0)}(k)$, that equals 1 for $k \leq \frac{5}{8}k_0$ and vanishes for $k \geq \frac{3}{4}k_0$. The positive number k_0 is just a reference wave number that fixes units. The argument of this template function is then dilated, setting $\phi_{(m)}(k) = \phi(2^{-m}k)$ and then the template is differenced, setting $\psi_{(0)}(k) = \phi_{(1)}(k) - \phi_{(0)}(k)$, and then $\psi_{(m)}(k) = \psi_{(0)}(2^{-m}k) = \phi_{(m+1)}(k) - \phi_{(m)}(k)$ for all integers m. The functions $\psi_{(m)}(k)$ are identically one for $k \in [\frac{3}{4}2^m k_0, \frac{5}{4}2^m k_0]$, and vanish outside the interval $[\frac{5}{8}2^m k_0, \frac{3}{2}2^m k_0]$. The relationship $1 = \phi_{(m)}(\mathbf{k}) + \sum_{n=m}^{\infty}\psi_n(\mathbf{k})$ holds for any integer m. One defines the Littlewood-Paley operators $S^{(m)}$ and Δ_n as multiplication, in Fourier representation, by $\phi_{(m)}(\mathbf{k})$ and, respectively by $\psi_{(n)}(\mathbf{k})$. Symbolically this means that the identity operator is written as $I = S^{(m)} + \sum_{n=m}^{\infty}\Delta_n$. The Littlewood-Paley decomposition of a function F is $F = S^{(m)}F + \sum_{n=m}^{\infty}\Delta_n F$. For mean-zero functions that decay at infinity, the terms $S^{(m)}$ becomes negligible when $m \to -\infty$ and therefore, for such functions one can write $F = \sum_{n=-\infty}^{\infty}\Delta_n F$. It is easy to see that for each fixed $k > 0$ at most three Δ_n do not vanish in their Fourier representation at k (i.e. the conditions $k \in [\frac{5}{8}2^n k_0, \frac{3}{2}2^n k_0]$ can be satisfied by at most three integers n, because $n \in [-1 + \log_2(\frac{k}{k_0}), 1 + \log_2(\frac{k}{k_0})]$). The operators $S^{(m)}$ and Δ_n can be viewed as convolution operators. In particular, for every $n = \pm 1, \pm 2, \ldots$, $\Delta_n = \int d\mathbf{h}\Psi_{(n)}(\mathbf{h})\delta_{\mathbf{h}}$. Here $\Psi_{(n)}$ is the function whose Fourier transform is $\psi_{(n)}$, $\widehat{\Psi_{(n)}} = \psi_{(n)}$, and $\delta_{\mathbf{h}}$ is the finite difference operator, $(\delta_{\mathbf{h}}F)(\mathbf{r}) = F(\mathbf{r} - \mathbf{h}) - F(\mathbf{r})$. Thus Δ_n is a weighted sum of finite difference operators at scale $2^{-n}k_0^{-1}$ in physical space, (k_0^{-1} provides thus an (arbitrary) length unit).

The Littlewood-Paley decomposition of the solutions of the quasi-geostrophic equation is performed at each instance of time,

$$\theta(\mathbf{r}, t) = \sum_{n=-\infty}^{\infty}\theta_{(n)}(\mathbf{r}, t).$$

We wrote for ease of notation $\theta_{(n)}$ instead of $\Delta_n\theta$, thus,

$$\theta_{(n)}(\mathbf{r},t) = \int d\mathbf{h}\Psi_{(n)}(\mathbf{h})\delta_{\mathbf{h}}(\theta)(\mathbf{r},t).$$

The finite difference is taken at equal times, $\delta_{\mathbf{h}}(\theta)(\mathbf{r},t) = \theta(\mathbf{r}-\mathbf{h},t)-\theta(\mathbf{r},t)$. There is an analogous decomposition for the velocity $\mathbf{v}$ and the forcing term f. In particular, in Fourier variables, the Littlewood-Paley components of $\mathbf{v}$ are given by $\widehat{\mathbf{v}_{(n)}}(\mathbf{k},t) = \psi_{(n)}(\mathbf{k})\widehat{\mathbf{v}}(\mathbf{k},t)$. The Littlewood-Paley spectrum [6] is

$$E_{LP}(k) = \frac{1}{k}\sum_{-1+\log_2(\frac{k}{k_0})\le n\le 1+\log_2(\frac{k}{k_0})} \langle|\widehat{\mathbf{v}_{(n)}}(\mathbf{k},t)|^2\rangle \tag{5}$$

where $\langle\dots\rangle$ is space-time average. The relation to the usual energy spectrum is straighforward. Because

$$\widehat{\mathbf{v}}(\mathbf{k},t) = \sum_{-1+\log_2(\frac{k}{k_0})\le n\le 1+\log_2(\frac{k}{k_0})} \widehat{\mathbf{v}}_{(n)}(\mathbf{k},t), \tag{6}$$

it follows that the usual energy spectrum

$$E(k) = \frac{1}{k}\langle|\widehat{\mathbf{v}}(\mathbf{k},t)|^2\rangle \tag{7}$$

satisfies

$$E(k) \le 3E_{LP}(k). \tag{8}$$

Clearly, because the functions $\psi_{(n)}$ are non-negative and bounded by 1, one also has

$$E(k) \ge \frac{1}{3}E_{LP}(k). \tag{9}$$

The temporal evolution of the system induces an evolution

$$\partial_t\theta_{(n)} + \mathbf{v}\cdot\nabla\theta_{(n)} + w_E\Lambda\theta_{(n)} = R_{(n)} \tag{10}$$

where

$$R_{(n)} = f_{(n)} + \int d\mathbf{h}\Psi_{(n)}(\mathbf{h})\nabla_{\mathbf{h}}\cdot(\delta_{\mathbf{h}}(\mathbf{v})\delta_{\mathbf{h}}\theta) \tag{11}$$

and $f_{(n)}$ is the Littlewood-Paley component of the forcing term. Multiplying (10) with $\theta_{(n)}$ and taking space-time average, one obtains the balance

$$w_E k \langle \left| \widehat{\theta_{(n)}}(\mathbf{k}, t) \right|^2 \rangle = \langle R_{(n)}(\mathbf{r}, t) \theta_{(n)}(\mathbf{r}, t) \rangle. \tag{12}$$

Let us consider the case when the forcing term has a limited support in Fourier space, $\widehat{f}(\mathbf{k}, t) = 0$ for $k \notin [k_a, k_b]$, with $0 < k_a < k_b < \infty$. The inverse cascade region will be described by wave numbers smaller than the minimal injection wave number k_a. The inverse cascade region corresponds thus, in the Littlewood-Paley decomposition, to indices $n > -\infty$ that satisfy $2^{n+1} k_0 < k_a$. We show now that the right hand side of the equation (12) is bounded above, uniformly for all such $n > -\infty$. Because we are in a region where $f_{(n)} = 0$, the term in the right-hand side of (12) can be written, after one integration by parts, as

$$\langle R_{(n)}(\mathbf{r}, t) \theta_{(n)}(\mathbf{r}, t) \rangle =$$

$$- \int d\mathbf{h} \nabla_{\mathbf{h}} \Psi_{(n)}(\mathbf{h}) \langle \delta_{\mathbf{h}}(\mathbf{v})(\mathbf{r}, t) \delta_{\mathbf{h}}(\theta)(\mathbf{r}, t) \theta_{(n)}(\mathbf{r}, t) \rangle. \tag{13}$$

This is a weighted sum of triple correlations. We will analyze each of the three terms involved in it, in an elementary but rigorous fashion. Because we aim at an upper bound, we will not try to optimize the prefactors. Using the Fourier inversion formula

$$\theta_{(n)}(\mathbf{r}, t) = (2\pi)^{-2} \int d\mathbf{k} e^{i\mathbf{r}\cdot\mathbf{k}} \psi_{(n)}(\mathbf{k}) \widehat{\theta}(\mathbf{k}, t),$$

the term $\theta_{(n)}$ is bounded pointwise by applying the Schwartz inequality:

$$|\theta_{(n)}(\mathbf{r}, t)| \leq (2\pi)^{-2} \|\psi_{(n)}\| \|\widehat{\theta}\|, \tag{14}$$

with $\| \cdots \|$ the mean square norm. Using the fact that $\psi_{(n)}$ is a dilate of $\psi_{(0)}$, we get

$$|\theta_{(n)}(\mathbf{r}, t)| \leq c_\psi 2^n E(t)^{\frac{1}{2}}, \tag{15}$$

where $c_\psi^2 = (2\pi)^{-2} \int d\mathbf{k} |\psi_{(0)}(\mathbf{k})|^2$ and $E(t) = \int d\mathbf{r} |\theta(\mathbf{r}, t)|^2 = \int d\mathbf{r} |\mathbf{v}(\mathbf{r}, t)|^2$ is the instaneous total energy. In the equality above we used Plancherel's

identity $\|F\| = (2\pi)^{-1}\|\widehat{F}\|$, and (4). In order to bound the other two terms we note that, from Plancherel, we have

$$\int d\mathbf{r}\, |\delta_{\mathbf{h}}\theta(\mathbf{r},t)|^2 = (2\pi)^{-2}\int d\mathbf{k}\, \left|e^{-i\mathbf{h}\cdot\mathbf{k}} - 1|^2\right| \widehat{\theta}(\mathbf{k},t)|^2.$$

Using $|e^{-i\mathbf{h}\cdot\mathbf{k}} - 1|^2 \le 4hk$, we deduce

$$\int d\mathbf{r}\, |\delta_{\mathbf{h}}\theta(\mathbf{r},t)|^2 \le 4h\eta(t), \tag{16}$$

where $\eta(t) = \int d\mathbf{r}\theta(\mathbf{r},t)\Lambda\theta(\mathbf{r},t)$. The term involving $\delta_{\mathbf{h}}\mathbf{v}$ is bounded using the same argument. In view of (4), the bound is by the same quantity:

$$\int d\mathbf{r}\, |\delta_{\mathbf{h}}\mathbf{v}(\mathbf{r},t)|^2 \le 4h\eta(t), \tag{17}$$

Putting the three inequalities (15, 16, 17) together with the Schwartz inequality, we deduce that the triple correlation term that is integrated in (13) obeys

$$\left|\langle\delta_{\mathbf{h}}(\mathbf{v})(\mathbf{r},t)\delta_{\mathbf{h}}(\theta)(\mathbf{r},t)\theta_{(n)}(\mathbf{r},t)\rangle\right| \le 4c_{\psi}2^n h\langle E(t)^{\frac{1}{2}}\eta(t)\rangle \tag{18}$$

In view of the fact that the functions $\Psi_{(n)}$ are dilates of a fixed function, we deduce that

$$|\langle R_{(n)}(\mathbf{r},t)\theta_{(n)}(\mathbf{r},t)\rangle| \le 2^{n+1}C_{\psi}\eta E^{\frac{1}{2}}. \tag{19}$$

Here E is the maximum total (not per unit volume) kinetic energy on the time interval, $E = \sup_t E(t)$. The constant

$$C_{\psi} = 2c_{\psi}\int d\mathbf{h} h|\nabla_{\mathbf{h}}\Psi_{(0)}| = c_0 k_0 \tag{20}$$

is proportional to k_0 and depends on the choice of the Littlewood-Paley template $\psi_{(0)}$ only through the non-dimensional positive absolute constant c_0. The number

$$\eta = \langle\eta(t)\rangle$$

is related to the long time dissipation. It can be bound in terms of the forcing term using the total balance

$$\frac{1}{2}\frac{d}{dt}E(t) + w_E\eta(t) = \int d\mathbf{r} f(\mathbf{r},t)\theta(r,t),$$

which follows from (1) after multiplication by θ and integration.Writing the integral in Fourier variables and using the fact that the support of the forcing is bounded below by $k_a > 0$ one obtains the bound

$$\eta \leq w_E^{-2} k_a^{-1} \langle |f(\mathbf{r}, t)|^2 \rangle. \tag{21}$$

This bound diverges for very large scale forcing, i.e. when $k_a \to 0$. Nevertheless, because of the presence of the coefficient 2^{n+1} in (19) and the fact that $2^{n+1} k_0 \leq k_a$ in the inverse cascade region, the total bound on the spectrum does not diverge as $k_a \to 0$: inserting (21) in (19) and using (20) we get

$$|\langle R_{(n)}(\mathbf{r}, t) \theta_{(n)}(\mathbf{r}, t) \rangle| \leq c_0 w_E^{-2} E^{\frac{1}{2}} \langle |f(\mathbf{r}, t)|^2 \rangle. \tag{22}$$

Now, using (22) in (12) and recalling the definition (5) and the inequality (8) we obtain

$$E(k) \leq C k^{-2} \tag{23}$$

for all $k < k_a$. This is the main result of this letter. The constant has units of length per time squared and is given by

$$C = 3 c_0 E^{\frac{1}{2}} w_E^{-3} \langle |f(\mathbf{r}, t)|^2 \rangle. \tag{24}$$

The upper bound proved in this letter holds in greater generality than presented here. First of all, the spectrum of the forces need not be confined to the band $[k_a, k_b]$. The role played by $(k_a)^{-1}$ is then played by the ratio $\langle k^{-1} |\widehat{f}(\mathbf{k}, t)|^2 \rangle \{ \langle |\widehat{f}(\mathbf{k}, t)|^2 \rangle \}^{-1}$. Secondly, the results and methods apply to a much wider class of quasi-geostrophic equations. In fact, the quasi-geostrophic equation chosen here is the simplest version of a class of equations in which the potential vorticity q is advected by a a three dimensional velocity field that can be derived from a stream function ψ (not to be confused with our Littlewood-Paley cutoff functions). The velocity has no vertical component $\mathbf{v} = (u, v) = (-\partial_y \psi, \partial_x \psi)$. The potential vorticity q is advected following horizontal trajectories, $\partial_t q + \mathbf{v} \cdot \nabla q + \beta v = F$, where F includes sources and damping. The potential vorticity and the stream function are functions of three space variables (x, y, z); the relation $q = (\partial_{xx} + \partial_{yy} + \partial_{zz}) \psi$ closes this equation. The boundary conditions at $z = 0$ are (1) with $\theta = \partial_z \psi$. One decomposes $\psi = \psi_B + \psi_N$ in a sum of a harmonic function ψ_B, $(\partial_{xx} + \partial_{yy} + \partial_{zz}) \psi_B = 0$, $\partial_z \psi_{B|z=0} = \theta$ and a function that satisfies homogeneous Neumann boundary conditions, $(\partial_{xx} + \partial_{yy} + \partial_{zz}) \psi_N = q$,

$\partial_z \psi_{N|z=0} = 0$. The ensuing equations on the boundary can be analyzed as above, using properties of the smooth evolution of q.

Two main ingredients were used in the proof. The first one is the way in which the spectrum of the potential temperature is related to the energy spectrum. The second, and the essential ingredient is the fact that the relaxation time at wave number k is roughly $(w_E k)^{-1}$ in the range of wave numbers considered. This dependence is an important by-product of the quasi-geostrophic model. In contradistinction with direct cascade models where there is a dissipation anomaly, in the quasi-geostrophic models the coefficient w_E is not vanishingly small. Moreover, the physical forcing amplitude obtained from an Ekman boundary layer is proportional to w_E. The explicit presence of the large scale term $E^{\frac{1}{2}}$ in the prefactor C is a reflection of the fact that the k^{-2} spectrum is modified near $k = 0$. The same is true for the Kolmogorov-Kraichnan spectrum: the integrals diverge and require an infrared cutoff. Such a cutoff can be achieved mathematically in two different ways. One may impose a smallest wave number k_{min} and boundary conditions; or one can modifiy the dissipation law so that, in the limit $k \to 0$ one has a finite relaxation time. In either case one can prove an *a priori* upper bound on E depending on the forcing and dissipation mechanism. The fact that the large scales are nearly conservative, with finite energy, was used in [2] to compare the k^{-2} and $k^{-\frac{5}{3}}$ spectra with the same largest scale energy. Our upper bound (23) confirms theoretically the separation of the two spectra when the injection length scales is small enough. This is indeed the case in the experiment [2]: forcing was applied through 120 holes of diameters of .25 cm, some 8 times larger than the Ekman boundary layer length.

In summary, we have proved that the energy spectrum of a forced surface quasi-geostrophic equation is bounded above by Ck^{-2} for wave numbers that are smaller than the force's injection wave number. Such a bound distinguishes the quasi-geostrophic model from a two-dimensional Navier-Stokes model, and agrees with the recent experimental evidence of [2].

The author acknowledges useful discussions with R. T. Pierrehumbert and with H.L. Swinney. This research is supported in part by NSF DMS-0202531 and by the ASCI Flash Center at the University of Chicago under DOE contract B341495.

References

[1] U. Frisch, *Turbulence*, Cambridge U. Press, Cambridge (1995).

[2] Ch. N. Baroud, B.B. Plapp, Z-S She, H. L. Swinney, Phys. Rev. Lett. **88**, 114501-1 (18 March 2002).

[3] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag,New York, (1987); I.M. Held, R.T. Pierrehumbert, S.T. Garner, K.L Swanson, J. Fluid Mech. **282** 1 (1995); R. T. Pierrehumbert, K.L. Swanson, Annu. Rev. Fluid Mech. **27**419 (1995)

[4] P. Constantin SIAM Review **36** 73 (1994), P. Constantin, A. Majda, E. Tabak, Nonlinearity **7** 1495 (1994), P. Constantin, Q. Nie, N. Schorghoffer, Phys. Lett. A **241** 168, (27 April 1998), P. Constantin, J. Wu, SIAM J. Math. Anal. **30** 937 (1999), P. Constantin, D. Cordoba, J. Wu, Indiana U. Math. J. **50** 97, (2001).

[5] M. E. Taylor, *Pseudodifferential Operators and Nonlinear PDE*,Birkhauser, Boston (1991).

[6] P. Constantin, Theor. Comp. Fluid Dyn. **9** 182 (1997), P. Constantin, Q. Nie, S. Tanveer, Phys. Fluids **11** 2251 (1999).